



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

A NECESSARY CONDITION THAT ALL THE ROOTS OF AN ALGEBRAIC EQUATION BE REAL

BY OLIVER D. KELLOGG

IN the ANNALS OF MATHEMATICS for July 1903, Professor Van Vleck has given a sufficient condition that an algebraic equation with real coefficients

$$(1) \quad c_0 + c_1x + c_2x^2 + \dots + c_nx^n = 0$$

have the maximum number of imaginary roots, that is, n or $n - 1$ according as n is even or odd. While the condition is not so stated, it appears from the proof that it is sufficient if the coefficients of even index, c_{2i} , and the determinants

$$\begin{vmatrix} c_{2i} & , & c_{2i+1} \\ c_{2i+1} & , & c_{2i+2} \end{vmatrix},$$

are all positive. It is interesting to remark that if any one of the determinants,

$$\begin{vmatrix} c_{j-1} & , & c_j \\ c_j & , & c_{j+1} \end{vmatrix},$$

among which the above are found, is positive, or if any vanishes for $k < j < n$, where c_k is the first coefficient of (1) different from zero, then the equation is sure to have *some* imaginary roots. This fact is merely another aspect of the following theorem:

If all the roots of the equation (1) are real, all the determinants

$$D_j = \begin{vmatrix} c_{j-1} & , & c_j \\ c_j & , & c_{j+1} \end{vmatrix} \quad (k < j < n)$$

are negative.

This is a result of a well known corollary to Des Cartes' Rule of Signs, to the effect that if all the roots of an algebraic equation are real, and if any intermediate coefficient is zero, the adjacent coefficients must be different from

zero and have opposite signs. From this it follows at once that if any c_j vanishes, the corresponding $D_j = c_{j-1} c_{j+1} < 0$. Moreover, it is evident that $D_k = -c_k^2 < 0$, this inequality holding even for $k = 0$ if we define c_{-1} as zero.

If c_j is different from zero, introduce into the equation (1) a new real root a , obtaining

$$(2) \quad (c_0 a + c_{-1}) + (c_1 a + c_0) x + (c_2 a + c_1) x^2 + \cdots + c_n x^n = 0.$$

Then choosing a so that the coefficient $c_j a + c_{j-1} = 0$ and applying the corollary cited to the equation (2) we find

$$\left(-\frac{c_{j-1}^2}{c_j} + c_{j-2}\right) \left(-\frac{c_{j+1}c_{j-1}}{c_j} + c_j\right) < 0,$$

that is,

$$D_{j-1} D_j > 0,$$

so that D_j and D_{j-1} are different from zero and have the same signs. Hence all the determinants in the series have the same sign, and since D_k is negative, it follows that D_j is negative for all values of j as was to be proved.

If instead of forming equation (2) we form from (1) an equation with two new real roots, and express the condition that two successive coefficients of the new equation cannot vanish, we shall be led to the theorem:

If all the roots of the equation (1) are real, none of the quadratics

$$\begin{vmatrix} x^2 & , & x & , & 1 \\ c_{j-1} & , & c_j & , & c_{j+1} \\ c_j & , & c_{j+1} & , & c_{j+2} \end{vmatrix} = 0 \quad (k \leq j < n, \quad c_{-1} = 0, \quad c_{n+1} = 0)$$

has real roots.

COLUMBIA, MISSOURI,
JULY, 1907.